Laplace's Equation

Heat eq: $u_t = k u_{xx}$ (1-D) $\qquad$ $u_t = k(u_{xx} + u_{yy})$ (2-D)

$x=0$ $\qquad$ $x=L$

$y=b$

$y=0$

$x=0$ $\qquad$ $x=a$

Wave eq: $u_{tt} = a^2 u_{xx}$ (1-D) $\qquad$ $u_{tt} = a^2(u_{xx} + u_{yy})$ (2-D)

the right side of them is : the second partial derivs with respect to space variables (x, y, z, etc)

they can all be represented by $\nabla^2 u$

Laplacian operator

$$\nabla^2 = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} + \cdots\right)$$

if $u = u(x,y)$

then $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

the value of $\nabla^2 u$ at a point gives us the shape information

$u_{xx} > 0 \rightarrow \nabla^2 u > 0 \rightarrow$ 

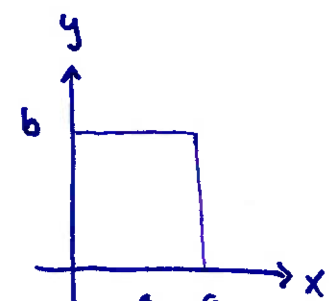
value of u here is lower than the average nearby

let's look at the 2D heat eq: $u_t = k\nabla^2 u$

$\rightarrow$ steady-state solution ($u_t = 0$)

we get $\nabla^2 u = 0 \rightarrow \boxed{u_{xx} + u_{yy} = 0}$ Laplace's Eq.

set up: $0 < x < a \qquad 0 < y < b$



$u(x,0) = f_1(x)$

4 BCs: one for each edge

$u(x,0) = f_1(x)$ lower edge

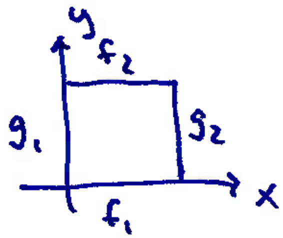
$u(x,b) = f_2(x)$ top edge

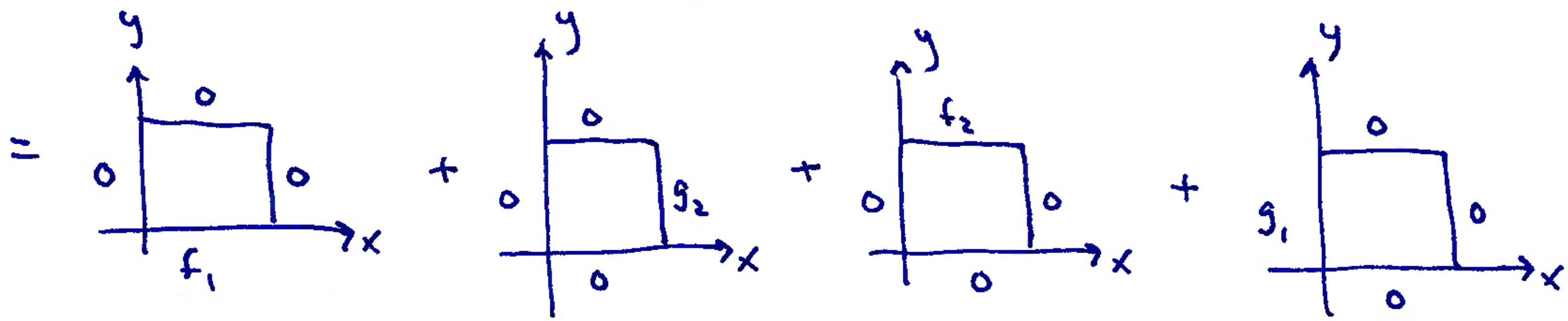
$u(0,y) = g_1(y)$ left edge

$u(a,y) = g_2(y)$ right edge

goal: find $u(x,y)$ satisfying $u_{xx} + u_{yy} = 0$ and ALL BCs the problem can be simplified by using the principle of superposition because Laplace's eq. is linear

so, general solution





→ make 3 BCs homogeneous (0), rotate which is nonhomogeneous

as an example, let's solve the 3rd case above

$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \quad 0 < y < b$

$u(x, 0) = 0$ (bottom)

$u(0, y) = 0$ (left)

$u(a, y) = 0$ (right)

$u(x, b) = f(x)$ (top)

we will use the <u>separation of variables</u> again

$$u(x,y) = X(x)\,Y(y) \qquad\qquad u_{xx} + u_{yy} = 0$$

$$u_{xx} = X''Y \qquad u_{yy} = XY''$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = -\lambda$$ (just like in heat/wave eqs)

ODEs that results : $X'' + \lambda X = 0$

$Y'' - \lambda Y = 0$

homogeneous BCs: $u(x,0) = 0 \rightarrow Y(0) = 0$

$u(0,y) = 0 \rightarrow X(0) = 0$

$u(a,y) = 0 \rightarrow X(a) = 0$

Solve for X or Y first whichever has complete BCs (NOT always X first as in heat/wave eqs) solve $X'' + \lambda X = 0$ $\quad X(0) = X(a) = 0$ $\quad$ in heat/wave eqs w/ $a = L$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{a^2}}$$

$n = 1, 2, 3, \ldots$

$$\boxed{X_n = \sin\left(\frac{n\pi}{a}x\right)}$$

now $Y'' - \lambda Y = 0$ $\quad Y(0) = 0$

$$Y'' - \frac{n^2\pi^2}{a^2}Y = 0$$

$$Y(y) = Ae^{\frac{n\pi y}{a}} + Be^{-\frac{n\pi y}{a}}$$

or

$$Y(y) = C_1 \cosh\left(\frac{n\pi y}{a}\right) + C_2 \sinh\left(\frac{n\pi y}{a}\right)$$

choose either whichever is more convenient w/ BC

here, the hyperbolic ones are better

$Y(0) = 0 \rightarrow C_1 = 0$ $\quad \boxed{Y_n = \sinh\left(\frac{n\pi y}{a}\right)}$

for each n, $u_n = X_n Y_n$

so, $$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)}$$

one last BC: $u(x,b) = f(x)$ (top)

so

$$f(x) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right)$$ sine series

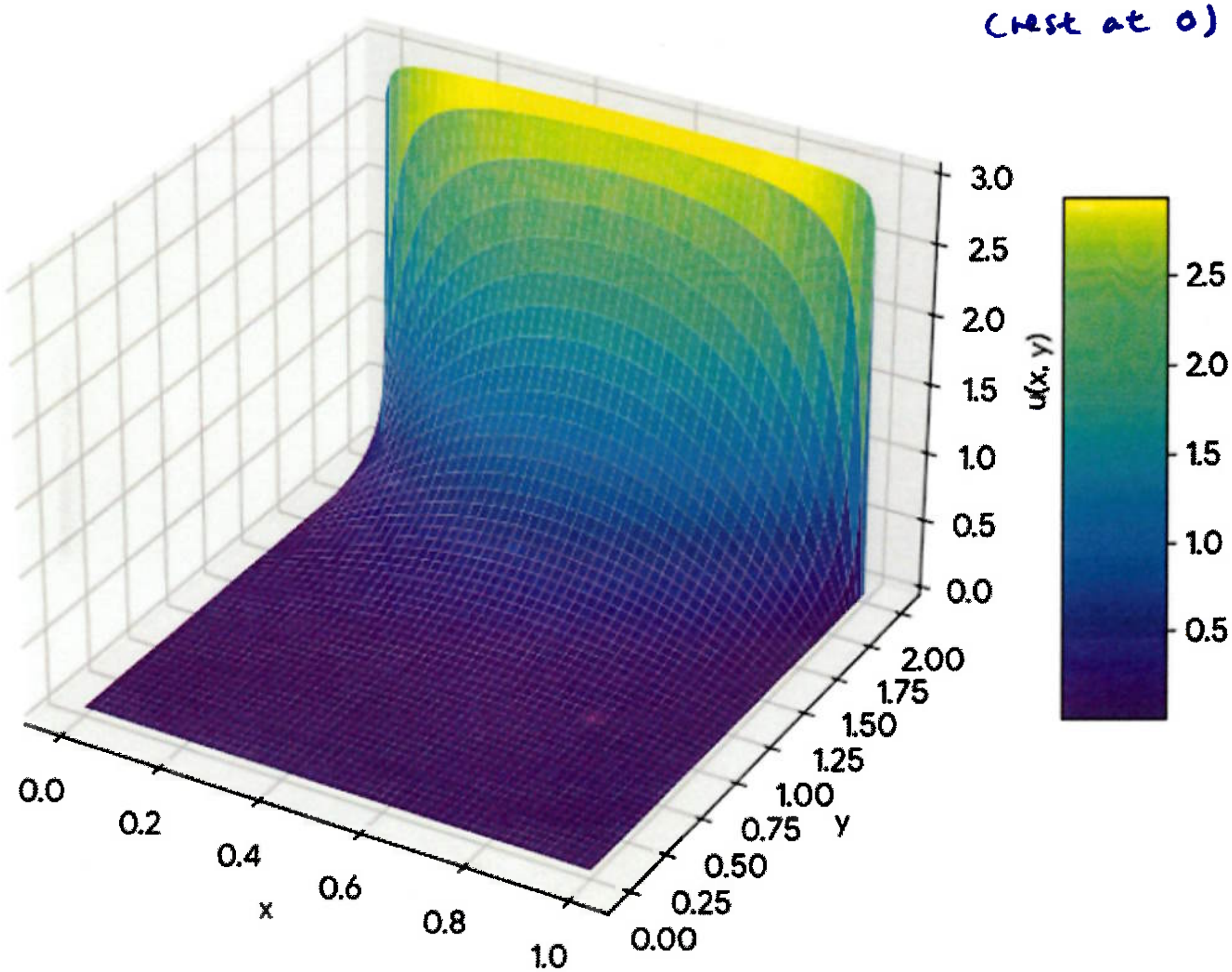
$$A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

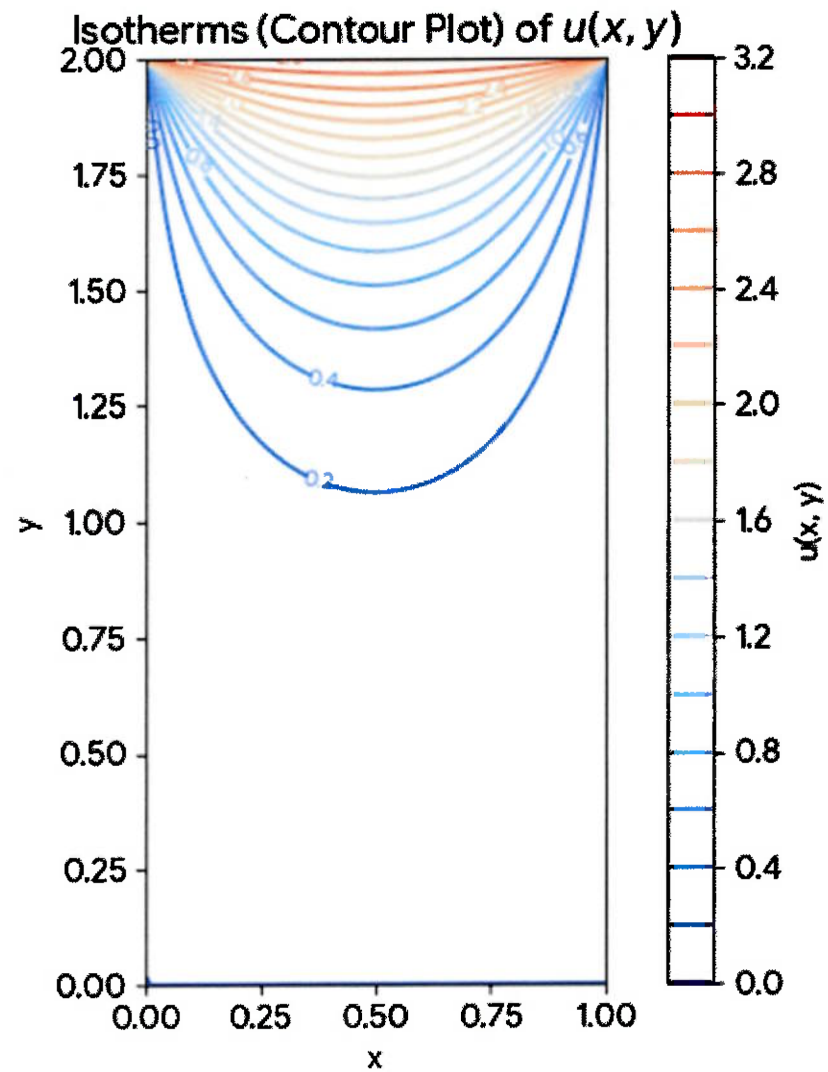
"L" for x because left side is $f(x)$

find A_n from that: $$\boxed{A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx}$$

Surface Plot of $u(x, y)$ (Laplace Equation Solution)

$a = 1$, $b = 2$,
top edge = 3
(rest at 0)





Isotherms (Contour Plot) of $u(x, y)$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.00
0.25
0.50
0.75
1.00
x
0.4
0.2
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0.0
u(x, y)